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Piecewise Coapproximation and the Whitney Inequality[1]

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All cases of the validity of the piecewise q-monotone analog of the Whitney inequality are clarified. For similar analogs of the Jackson inequality negative results are proved. © 2000 Academic Press

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1. INTRODUCTION

It is known that piecewise monotone analogues of the Whitney inequality sometimes are true are sometimes are false; see, e.g., [5, 12, 13] for the details. We are going to investigate the same problem here for the piecewise q-monotone case with $q>1$, in particular when $q=2$, for piecewise convex approximation. To this end we formulate four Whitney-type propositions and investigate all cases of their validity. We also prove some negative results for two Jackson-type propositions. For some other negative results see Zhou [15]. For the "pure" (that is not piecewise) q-monotone approximation with $q>1$, see, e.g., [12].

2. NOTATIONS AND STATEMENT OF THE MAIN RESULTS

2.1. *Notations*

Let $\mathbb{I} := [-1, 1]$; $\mathbb{C}^{(0)} := \mathbb{C}$ be the space of continuous functions f: $\mathbb{I} \to \mathbb{R}$, with the uniform norm

$$\|f\| = \max_{x \in \mathbb{I}} |f(x)|;$$

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let $\mathbb{P}_n$ be the space of algebraic polynomials of degree $\leqslant n$; and let

$$E_n(f) := \inf_{p_n \in \mathbb{P}_n} \|f - p_n\|$$

be the error of the best uniform approximation of $f \in \mathbb{C}$; $\mathbb{C}^{(r)} := \{f : f^{(r)} \in \mathbb{C}\}$, $r \in \mathbb{N}$.

For $s \in \mathbb{N}$ we denote by $\mathbb{Y}_s$ the set of all collections $Y := \{y_i\}_{i=1}^s$ of s distinct points y_i, such that $-1 < y_s < \cdots < y_1 < 1$. For each $Y = \{y_i\}_{i=1}^s \in \mathbb{Y}_s$ put

$$\Pi(x) := \Pi(x; Y) := \prod_{i=1}^{s} (x - y_i).$$

Set $\mathbb{Y} := \bigcup_{s=1}^{\infty} \mathbb{Y}_s$. Let $Y \in \mathbb{Y}$, $q \in \mathbb{N}$. For $f \in \mathbb{C}^{(q)}$ we will write $f \in \Delta^{(q)}(Y)$, iff

$$f^{(q)}(x)\, \Pi(x; Y) \geqslant 0, \qquad x \in \mathbb{I}.$$

For $f \in \mathbb{C}$ (not necessarily $f \in \mathbb{C}^{(q)}$) we will write $f \in \Delta^{(q)}(Y)$, iff for every $v = 0, ..., s$ and for each collection of $q+1$ points $z_{j,v} \in [y_{v+1}, y_v]$, $j = 0, ..., q$, the inequality

$$(-1)^v\, [z_{0,v}, ..., z_{q,v}; f] \geqslant 0$$

holds, where $y_0 := 1$, $y_{s+1} := -1$, and

$$[t_0, ..., t_m; f]$$

is the divided difference of order m of a function f at the knots $t_0, ..., t_m$. Evidently, when $f \in \mathbb{C}^{(q)}$, both definitions of $\Delta^{(q)}(Y)$ coincide. Note that $\Delta^{(1)}(Y)$ is the set of piecewise monotone functions on $\mathbb{I}$ and $\Delta^{(2)}(Y)$ is the set of piecewise convex functions on $\mathbb{I}$.

For $Y \in \mathbb{Y}$ and $f \in \Delta^{(q)}(Y)$ set

$$E_n^{(q)}(f; Y) := \inf_{p_n \in \Delta^{(q)}(Y) \cap \mathbb{P}_n} \|f - p_n\|,$$

the error of best uniform piecewise q-monotone approximation of f.

Finally denote by

$$\omega_k(f; t) := \sup_{h \in [0, t]} \max_{x \in [-1, 1-kh]} \left| \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+jh) \right|, \qquad t \geqslant 0,$$

the kth order modulus of continuity of a function $f \in \mathbb{C}$.

Everywhere below,

$$k \in \mathbb{N}, \qquad (r+1) \in \mathbb{N}, \quad s \in \mathbb{N}, \quad q \in \mathbb{N}.$$

2.2. *Whitney-Type Propositions*

For $f \in \mathbb{C}^{(r)}$ the Whitney [14] inequality

$$E_{k+r-1}(f) \leqslant c(k, r)\, \omega_k(f^{(r)}; 1) \tag{2.1}$$

is well known, where $c(k, r) = \text{const}$, depending only on k and r; see, e.g., (4.5) in [2, Chap. 6].

Here we formulate two Whitney-type propositions: the "strong" Proposition $W(k, r, s, q)$ and the "weak" Proposition $W(k, r, s, q, Y)$. Then we formulate two auxiliary propositions, the use of which will be discussed in the next Section 2.3. These four propositions sometimes are true, sometimes are false. In Theorem 2 we will clearly all cases where Whitney-type propositions are true or false. In Theorem 3 we will clearly the same for the auxiliary propositions. To illustrate Theorems 2 and 3 we formulate Theorem 1, which is a particular case, say the case ($s = 4$, $q = 6$).

Proposition $W(k, r, s, q)$. *There exists a constant $B = B(k, r, s, q)$ such that for each $Y \in \mathbb{Y}_s$ and $f \in \mathbb{C}^{(r)} \cap \Delta^{(q)}(Y)$ we have*

$$E^{(q)}_{k+r-1}(f; Y) \leqslant B \omega_k(f^{(r)}; 1). \tag{2.2}$$

Proposition $W(k, r, s, q, Y)$. *Let $Y \in \mathbb{Y}_s$. There exists a constant $B = B(k, r, s, q, Y)$ such that for each $f \in \mathbb{C}^{(r)} \cap \Delta^{(q)}(Y)$ the inequality* (2.2) *holds.*

Proposition $A(k, r, s, q)$. *Propositions $W(k, r, m, q)$ are true for all $m = 1, ..., s$.*

Proposition $A(k, r, s, q, Y)$. *Let $Y \in \mathbb{Y}_s$. Propositions $W(k, r, m, q, Y_m)$ are true for all $m = 1, ..., s$ and $Y_m \in \mathbb{Y}_m$ such that $Y_m \subseteq Y$.*

Theorem 1. *Let $q = 6$ and $s = 4$. The truth table of Propositions $W(k, r, s, q)$, $W(k, r, s, q, Y)$, $A(k, r, s, q)$ and $A(k, r, s, q, Y)$ has the form*

$$
\begin{array}{cccccccccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
 & + & + & + & + & + & + & + & + & + & + & + & + & \cdots \\
q+s & + & + & + & + & + & + & + & + & + & + & + & + & \cdots \\
 & + & + & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
 & + & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
 & + & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
q & + & \oplus & \oplus & \ominus & \ominus & \ominus & - & - & - & - & - & - & \cdots \\
 & + & \oplus & \ominus & \ominus & \ominus & - & - & - & - & - & - & - & \cdots \\
 & + & + & \oplus & \ominus & \ominus & \ominus & - & - & - & - & - & - & \cdots \\
 & + & + & + & \oplus & \ominus & \ominus & \ominus & - & - & - & - & - & \cdots \\
2 & + & + & + & + & \oplus & \ominus & \ominus & \ominus & - & - & - & - & \cdots \\
1 & + & + & + & + & + & \oplus & \ominus & \ominus & \ominus & - & - & - & \cdots \\
0 & + & + & + & + & + & + & \oplus & \ominus & \ominus & \ominus & - & - & \cdots \\
 & 1 & 2 & 3 & & & & & & & & & & k
\end{array}
$$

where "+" stands for the cases where Propositions $W(k, r, s, q)$ and $A(k, r, s, q)$ are true, and hence, for each $Y \in \mathbb{Y}_s$, Propositions $W(k, r, s, q, Y)$ and $A(k, r, s, q, Y)$ are true as well; "$\oplus$" stands for the cases where Propositions $W(k, r, s, q)$ and $A(k, r, s, q)$ are false, but Propositions $W(k, r, s, q, Y)$ and $A(k, r, s, q, Y)$ are true for each $Y \in \mathbb{Y}_s$; "$\ominus$" stands for the cases where Propositions $W(k, r, s, q)$ and $A(k, r, s, q)$ are false, Proposition $A(k, r, s, q, Y)$ is false for each $Y \in \mathbb{Y}_s$, but Proposition $W(k, r, s, q, Y)$ is true for each $Y \in \mathbb{Y}_s$; "$-$" stand for the cases, where Propositions $W(k, r, s, q, Y)$ and $A(k, r, s, q, Y)$ are false for each $Y \in \mathbb{Y}_s$, and hence Propositions $W(k, r, s, q)$ and $A(k, r, s, q)$ are false as well.

We break up all collections (k, r, s, q) into four types.

Definition 1. We will say that a collection (k, r, s, q) is of type "+" iff $(k=1)$, or $(k+r \leqslant q)$, or $(q+s \leqslant r)$, or $(r=q+s-1,\ k=2)$; "$\ominus$", iff $(r<q<r+k-1<q+s)$, or $(r=q,\ 3<k \leqslant s+2)$; "$-$", iff $(q+s-k<r<q)$, or $(r=q,\ k \geqslant s+3)$; "$\oplus$" in all other cases.

Theorem 2. *In the case of type "+" Proposition $W(k, r, s, q)$ is true; in all other cases Proposition $W(k, r, s, q)$ is false. In the cases of type "$-$" Proposition $W(k, r, s, q, Y)$ is false for each $Y \in \mathbb{Y}_s$; in all other cases Proposition $W(k, r, s, q, Y)$ is true for each $Y \in \mathbb{Y}_s$.*

For $q=1$ Theorem 2 is known; see, e.g., [5, 12, 13]. For $q>1$ Theorem 2 follows from Lemmas 3.2, 3.5, 4.1, and 4.2 below.

THEOREM 3. *In the cases of type "+" Proposition $A(k, r, s, q)$ is true; in all other cases Proposition $A(k, r, s, q)$ is false. In the cases of types "$\ominus$" and "$-$" Proposition $A(k, r, s, q, Y)$ is false for each $Y \in \mathbb{Y}_s$; in all other cases Proposition $A(k, r, s, q, Y)$ is true for each $Y \in \mathbb{Y}_s$.*

We shall not prove Theorem 3, since Theorem 3 is a trivial corollary of Theorem 2.

2.3. *Jackson-Type Propositions*

Everywhere below $n \in \mathbb{N}$.

PROPOSITION $J(k, r, s, q)$. *There exist two constants $B = B(k, r, s, q)$ and $N = N(i, r, s, q)$ such that for each $Y \in \mathbb{Y}_s$, $f \in \mathbb{C}^{(r)} \cap \Delta^{(q)}(Y)$, and $n \geqslant N$ we have*

$$E_n^{(q)}(f; Y) \leqslant B \frac{1}{n^r} \omega_k \left(f^{(r)}; \frac{1}{n}\right). \tag{2.3}$$

PROPOSITION $J(k, r, s, q, Y)$. *Let $Y \in \mathbb{Y}_s$. There exist two constant $B = B(k, r, s, q, Y)$ and $N = N(k, r, s, q, Y)$ such that for each $f \in \mathbb{C}^{(r)} \cap \Delta^{(q)}(Y)$ and $n \geqslant N$ the inequality* (2.3) *holds.*

In Section 4 we will prove

THEOREM 4. *In the cases of types "$\oplus$", "$\ominus$", and "$-$" Propositions $J(k, r, s, q)$ is false. In the cases of types "$\ominus$" and "$-$" Proposition $J(k, r, s, q, Y)$ is false for each $Y \in \mathbb{Y}_s$.*

For $q = 1$ Theorem 4 is known; see [5, 12, 13]. For the case ($r = 0$, $k > q + 1$) Theorem 4 follows by Zhou [15]. The first part of Theorem 4 is Lemma 4.1 below, the second part is Lemma 4.3.

Theorems 3 and 4 readily imply

THEOREM 5. *If Proposition $A(k, r, s, q)$ ($A(k, r, s, q, Y)$) is false, then Proposition $J(k, r, s, q)$ ($J(k, r, s, q, Y)$) is false as well*

Remark 1. For $q = 1$, Propositions $A(k, r, s, q)$ and $J(k, r, s, q)$ are equivalent; the same is true for Propositions $A(k, r, s, q, Y)$ and $J(k, r, s, q, Y)$. This follows by Newman [10], Iliev [6], Beatson and Leviatan [1], Shvedov [13], and Dzyubenko *et al.* [4], [5].

Remark 2. About Jackson-type propositions with $q > 1$, the authors know only one positive result. Kopotun *et al.* [7] proved the truth of Proposition $J(k, r, s, q, Y)$ for the case ($k + r \leqslant 3$, $q = 2$); moreover, they proved, that it holds with B or N independent of Y.

2.4. *Some Relationships*

In the sequel we will have constants c that may depend only on k, r, q, s, or some of these parameters. They may differ in different occurrences, even in the same line. We will denote by B_Y, B_Y^*, B_Y^{**} positive constants that depend only on k, r, s, q, and Y.

We denote by $L(x, f; t_0, ..., t_m)$ the Lagrange polynomial of degree $\leqslant m$, that interpolates a function f at the points $t_0, ..., t_m$.

Without special references we will often use the following well-known relations. The reader may find these relations in the monograph of DeVore and Lorentz [2].

For the divided differences $[t_0, ..., t_m; f]$ we have (see [2, Chap. 4, (7.3), (7.7), and (7.4)])

$$[f_0, ..., t_m; f] = \frac{f(t_m) - L(t_m, f; t_0, ..., t_{m-1})}{(t_m - t_0) \cdots (t_m - t_{m-1})}.$$

If, for all $j = 1, ..., m$, $t_j > t_{j-1}$ and $f(t_j)\, f(t_{j-1}) < 0$, then

$$f(t_m)[t_0, ..., t_m; f] > 0.$$

If $t_j \in [a, b]$, $j = 0, ..., m$, and $f \in \mathbb{C}^{(m)}(a, b)$, then

$$[t_0, ..., t_m; f] = \frac{1}{m!} f^{(m)}(\theta), \qquad \theta \in (a, b).$$

For the kth modulus of continuity $\omega_k(f; t)$ of a function $f \in \mathbb{C}$ we have (see [2, Chap. 2, (7.5), (7.13), (7.12)]) if $r > k$, then

$$\omega_r(f; t) \leqslant 2^{r-k} \omega_k(f; t),$$

whence

$$\omega_k(f; t) \leqslant 2^k \kappa \, \|f\|.$$

If $f \in \mathbb{C}^r$, then

$$\omega_{r+k}(f; t) \leqslant t^r \omega_k(f^{(r)}; t),$$

and

$$\omega_r(f; t) \leqslant t^r \, \|f^{(r)}\|.$$

For each polynomial $p_n \in \mathbb{P}_n$ we have (see [2, Chapter 4, (1.2)]), Markov's inequality

$$\|p_n'\| \leqslant n^2 \, \|p_n\|.$$

Dzyadyk's inequality [3], (see also [2, Chapter 8, (2.15)]). For each $\gamma \in \mathbb{R}$,

$$\|p_n' \rho_n^{\gamma+1}\| \leqslant C(\gamma) \, \|p_n \rho_n^{\gamma}\|,$$

where $C(\gamma)$ depends only on γ, and

$$\rho_n(x) := \frac{1}{n^2} + \frac{1}{n} \sqrt{1 - x^2}.$$

For $f \in \mathbb{C}^{(r)}$ and its polynomial of best approximation P_n^*, Leviatan's inequality [8], (see also [2, Chapter 8, (4.17)]),

$$\|(f^{(r)} - P_n^{*r}) \, \rho_n^r\| \leqslant \frac{c}{n^r} E_{n-r}(f^{(r)}),$$

holds. This implies for each polynomial $p_n \in \mathbb{P}_n$,

$$\|(f^{(r)} - p_n^{(r)}) \, \rho_n^r\| \leqslant \frac{c}{n^r} E_{n-r}(f^{(r)}) + c \, \|f - p_n\|, \tag{2.4}$$

since $c \, \|(P_n^{*(r)} - p_n^{(r)}) \, \rho_n^r\| \leqslant \|P_n^* - p_n\| \leqslant 2 \, \|f - p_n\|$.

We will also use the well-known inequality, for $f \in \mathbb{C}$ and $[a, b] \subset \mathbb{I}$,

$$\|f\| \leqslant c(b-a)^{1-k} \, (E_{k-1}(f) + \max_{x \in [a, b]} |f(x)|), \tag{2.5}$$

which is a consequence of the simple estimate

$$\begin{aligned} |P_{k-1}^*(x)| &\leqslant |P_{k-1}^*(x) - f(x)| + |f(x)| \\ &\leqslant E_{k-1}(f) + \max_{t \in [a, b]} |f(t)| =: \lambda, \qquad x \in [a, b]. \end{aligned}$$

Indeed, by [2, Chapter 2, (2.10)],

$$\|f\| \leqslant \|f - P_{k-1}^*\| + \|P_{k-1}^*\| \leqslant E_{k-1}(f) + c\lambda(b-a)^{1-k} \leqslant c\lambda(b-a)^{1-k}.$$

3. POSITIVE RESULTS

Lemma 3.1. *Let $Y \in \mathbb{Y}_s$ and $k \leqslant q$. If $f \in \Delta^{(q)}(Y)$, then*

$$E_{k-1}^{(q)}(f; Y) \leqslant c\omega_k(f; 1),$$

where $c = c(k, 0)$ is the constant in Whitney inequality (2.1).

Proof. Since $k-1<q$, then $E^{(q)}_{k-1}(f, Y)=E_{k-1}(f)$, hence by (2.1)

$$E^{(q)}_{k-1}(f, Y)=E_{k-1}(f)\leqslant c(k, 0)\, \omega_k(f; 1). \quad \blacksquare$$

LEMMA 3.2. *In the cases of type "+" Proposition $W(k, r, s, q)$ is true.*

Proof. We prove Lemma 3.2 by induction on q. Recall, for $q=1$ Theorem 2 is valid, hence Lemma 3.2 is valid as well. Assume that Lemma 3.2 is valid for some number $q-1\geqslant 1$ and prove it for the number q. To this end we take a collection (k, r, s, q) of type "+". If $r=0$, then Lemma 3.2 follows from Lemma 3.1. So let $r\neq 0$. Then by Definition 1 the collection $(k, r-1, s, q-1)$ is of type "+" as well, and hence our assumption implies, that Proposition $W(k, r-1, s, q-1)$ is true. Therefore, for each $Y\in \mathbb{Y}_s$ and $f\in \mathbb{C}^{(r)}\cap \Delta^{(q)}(Y)$,

$$E^{(q-1)}_{k+r-2}(f', Y)\leqslant B(k, r-1, s, q-1)\, \omega_k(f^{(r)}; 1), \tag{3.1}$$

since evidently $f'\in \mathbb{C}^{(r-1)}\cap \Delta^{(q-1)}(Y)$. For each polynomial $p_{k+r-1}\in \mathbb{P}_{k+r-1}\cap \Delta^{(q)}(Y)$ we have

$$p'_{k+r-1}\in \mathbb{P}_{k+r-2}\cap \Delta^{(q-1)},$$

$$p_{k+r-1}-p_{k+r-1}(0)+f(0)=: \tilde{p}_{k+r-1}\in \mathbb{P}_{k+r-1}\cap \Delta^{(q)}(Y),$$

$$f(x)-\tilde{p}_{k+r-1}(x)=\int_0^x (f'(u)-p'_{k+r-1}(u))\, du,$$

whence

$$E^{(q)}_{k+r-1}(f, Y)\leqslant E^{(q-1)}_{k+r-2}(f', Y).$$

This inequality and (3.1) imply the truth of Proposition $W(k, r, s, q)$, with a constant $B(k, r, s, q)\leqslant B(k, r-1, s, q-1)$. $\blacksquare$

LEMMA 3.3. *Let $Y\in \mathbb{Y}_s$, $q>1$ and $k=q+s$. If $f\in \Delta^{(q)}(Y)$, then*

$$E_{q-1}(f)\leqslant B_Y E_{k-1}(f).$$

Proof. Let us add to the points $y_1, ..., y_s$ some new points: put

$$y_0=: 1, \quad y_{s+1}:=y_s-\frac{y_s+1}{q-1}, \quad y_{s+2}:=y_s-2\,\frac{y_s+1}{q-1}, ..., y_{q+s-1}=-1.$$

Set

$$J_\nu:=(y_\nu, y_{\nu-1}), \quad \nu=1, ..., s+q-1; \quad L(x):=L(x; f; y_0, ..., y_{q+s-1}).$$

Note that there exists a least one number ν^* such that

$$L^{(q)}(x)\,\Pi(x) \leqslant 0 \tag{3.2}$$

for all $x \in J_{\nu^*}$. Indeed, otherwise the derivative $L^{(q)}(x)$ would change the sign at least s times, but $\deg L^{(q)}(x) \leqslant s-1$.

Now let us divide the interval J_{ν^*} by $q+2$ equidistant points $t_0 = y_{\nu^*}, \ldots, t_{q+1} = y_{\nu^*-1}$. Put

$$p_{q-1}(x) := L(x; f; t_1, \ldots, t_q), \qquad g(x) := f(x) - p_{q-1}(x).$$

Since $f \in \Delta^{(q)}(Y)$, then $[x, t_1, \ldots, t_q; f]\,\Pi(x) \geqslant 0$, $x \in J_{\nu^*}$, hence for $x \in J_{\nu^*}$ we get

$$g(x)\,\Pi(x) \prod_{j=1}^{q} (x - t_j) = \prod_{j=1}^{q} (x - t_j)^2\,[t_1, \ldots, t_q, x; f]\,\Pi(x) \geqslant 0. \tag{3.3}$$

Put

$$L_*(x) := L(x; g; y_0, \ldots, y_{q+s-1}); \qquad T_j := (t_j, t_{j+1}), \qquad j = 0, \ldots, q.$$

Then there is a number j_* such that for $x \in T_{j_*}$.

$$L_*(x)\,\Pi(x) \prod_{j=1}^{q} (x - t_j) \leqslant 0. \tag{3.4}$$

For otherwise $q+1$ points $\theta_j \in T_j$ exist, such that $\theta_j > \theta_{j-1}$, $L_*(\theta_j)\,L_*(\theta_{j-1}) < 0$, $j = 1, \ldots, q$, and $L_*(\theta_q)\,\Pi(\theta_q) > 0$, therefore

$$0 < [\theta_0, \ldots, \theta_q; L_*]\,\Pi(\theta_q) = \frac{1}{q!}\,L_*^{(q)}(\theta)\,\Pi(\theta_q) \tag{3.5}$$

for some $\theta \in J_\nu^*$, but since $L_*^{(q)} \equiv L^{(q)}$ and $\Pi(\theta_q)\,\Pi(\theta) > 0$, then (3.5) contradicts (3.2).

It follows from (3.3) and (3.4) that

$$g(x)\,L_*(x) \leqslant 0, \qquad x \in T_{j_*}, \tag{3.6}$$

therefore one can write

$$|g(x)| \leqslant |g(x) - L_*(x)| = |f(x) - L(x)|, \qquad x \in T_{j_*}.$$

Denote by $|J|$ the length of the shortest among intervals J_ν, $\nu = 1, \ldots, k-1$. Then, for each polynomial $P_{k-1} \in \mathbb{P}_{k-1}$ we have

$$\begin{aligned}|f(x)-L(x)| &= |(f(x)-P_{k-1}(x))-L(x, f-P_{k-1}; y_0, ..., y_{k-1})| \\ &\leqslant \|f-P_{k-1}\| \left(1+k\left(\frac{2}{|J|}\right)^{k-1}\right) \\ &=: B_Y^* \|f-P_{k-1}\|, \qquad x \in \mathbb{I},\end{aligned}$$

hence

$$\|f-L\| \leqslant B_Y^* E_{k-1}(f),$$

whence

$$|g(x)| \leqslant B_Y^* E_{k-1}(f), \qquad x \in T_{j_*}. \tag{3.7}$$

Since the length of T_{j_*} is greater than a constant B_Y^{**}, then (3.7) and (2.5) yield

$$\|g\| \leqslant B_Y E_{k-1}(f).$$

Thus

$$E_{q-1}(f) \leqslant \|f-p_{q-1}\| = \|g\| \leqslant B_Y E_{k-1}(f). \quad \blacksquare$$

LEMMA 3.4. *Let $Y \in \mathbb{Y}_s$, $q>1$ and $k \leqslant q+s$. If $f \in \Delta^{(q)}(Y)$, then*

$$E_{k-1}^{(q)}(f; Y) \leqslant B_Y \omega_k(f; 1).$$

Proof. For $k \leqslant q$ Lemma 3.4 follows from Lemma 3.1. For $q<k \leqslant q+s$ Lemma 3.4 follows from Lemma 3.3, Whitney inequality (2.1) and obvious relationships

$$E_{q+s-1}(f) \leqslant E_{k-1}(f), \qquad E_{k-1}^{(q)}(f; Y) \leqslant E_{q-1}^{(q)}(f; Y) = E_{q-1}(f). \quad \blacksquare$$

LEMMA 3.5. *In the case of type "+", "$\oplus$" and "$\ominus$" Proposition $W(k, r, s, q, Y)$ is true for each $Y \in \mathbb{Y}_s$.*

One proves Lemma 3.5 in the same way as Lemma 3.2, applying Lemma 3.4 instead of Lemma 3.1.

4. NEGATIVE RESULTS

4.1. *Cases* "$\oplus$"

We will use the arguments from [5].

EXAMPLE 4.1. For every n and $A > 0$, and for each q, s, and $q-1 \leqslant r \leqslant q+s-2$, there is a collection $Y(n, r, A, s) =: Y \in \mathbb{Y}_s$ and a function $f_{n, r, A} =: f \in \mathbb{C}^{(r)} \cap \Delta^{(q)}(Y)$ such that

$$E_n^{(q)}(f; Y) \geqslant A\omega_2(f^{(r)}; 1) \geqslant A 2^{-k+2} \omega_k(f^{(r)}; 1), \qquad k \geqslant 2. \qquad (4.1)$$

Proof. Without any loss of generality assume $n \geqslant r+1$. We take $b \in (0, 1)$ so that

$$\frac{1}{4bn^{2(r+1)}} - \frac{b^r}{4(r+1)!} = A,$$

and fix an arbitrary collection Y of points y_i such that $-1+b = y_1 > y_2 > \cdots > y_s > -1$. Set

$$Q_{r+1}(x) := (x - y_1)^{r+1};$$

$$f(x) := (x - y_1)_+^{r+1} := \begin{cases} Q_{r+1}(x), & \text{if } \; x \geqslant -1+b, \\ 0 & \text{if } \; x < -1+b. \end{cases}$$

Obviously, $f \in \mathbb{C}^{(r)} \cap \Delta^{(q)}(Y)$. For an arbitrary polynomial $p_n \in \Delta^{(q)}(Y) \cap \mathbb{P}_n$ put

$$R_n(x) := Q_{r+1}(x) - p_n(x)$$

and consider the divided difference $[y_1, ..., y_{r+2-q}; R_n^{(q)}]$. Since $p_n \in \Delta^{(q)}(Y)$, then $p_n^{(q)}(y_i) = 0$, $i = \overline{1, r+2-q}$, whence $[y_1, ..., y_{r+2-q}; p_n^{(q)}] = 0$. Besides, clearly,

$$[y_1, ..., y_{r+2-q}; Q_{r+1}^{(q)}] = \frac{(r+1)!}{(r+1-q)!},$$

i.e.

$$[y_1, ..., y_{r+2-q}; R_n^{(q)}] = \frac{(r+1)!}{(r+1-q)!}.$$

Hence there exists a point $\theta \in (-1, -1, +b)$ such that

$$R_n^{(r+1)}(\theta) = (r+1-q)! \, [y_1, ..., y_{r+2-q}; R_n^{(q)}] = (r+1)!.$$

Reasoning similarly to Lorentz and Zeller [9] (see also Shvedov [13]), we apply Markov inequality and get

$$\begin{aligned}(r+1)! = R_n^{(r+1)}(\theta) &\leqslant \|R_n\| \, n^{2(r+1)} \\ &\leqslant n^{2(r+1)}(\|f - p_n\| + \|f - Q_{r+1}\|) = n^{2(r+1)}(\|f - p_n\| + b^{r+1}),\end{aligned}$$

whence

$$\|f-p_n\| \geqslant \frac{(r+1)!}{n^{2(r+1)}} - b^{r+1}.$$

On the other hand,

$$\omega_2(f^{(r)}; 1) = \omega_2(f^{(r)} - Q_{r+1}^{(r)}; 1) \leqslant 2\, \|f^{(r)} - Q_{r+1}^{(r)}\| = 4(r+1)!\, b.$$

Therefore

$$\frac{\|f-p_n\|}{\omega_2(f^{(r)}; 1)} \geqslant \frac{1}{4bn^{2(r+1)}} - \frac{b^r}{4(r+1)!} = A. \quad \blacksquare$$

Remark. The corresponding example for $q=1$ was constructed by Shvedov [13].

COROLLARY. *For each q, $r<q$, s, n and $A>0$ there is a collection $Y(n, A, s, q) =: Y \in \mathbb{Y}_s$ and a function $f_{n, A, q} =: f \in \mathbb{C}^{(r)} \cap \Delta^{(q)}(Y)$ such that*

$$\begin{aligned} E_n^{(q)}(f; Y) &\geqslant A\omega_{q+1-r}(f^{(r)}; 1) \\ &\geqslant 2^{q+1-r-k} A\omega_k(f^{(r)}; 1), \qquad k \geqslant q+1-r. \end{aligned} \tag{4.2}$$

Indeed, for $r=q-1$ such function is constructed in Example 4.1; for $r<q-1$ one can take the same function and use the inequality $\omega_2(f^{(q-1)}; 1) \geqslant \omega_{q+1-r}(f^{(r)}; 1)$.

EXAMPLE 4.2. For every n and $A>0$, and for each s and q, there is a collection $Y(n, A, s, q) =: Y \in \mathbb{Y}_s$ and a function $f_{n, A, q}(x) =: f \in \mathbb{C}^{(r)} \cap \Delta^{(q)}(Y)$ such that

$$E_n^{(q)}(f; Y) \geqslant A\omega_3(f^{(r)}; 1) \geqslant A\, 2^{-k+3}\omega_k(f^{(r)}; 1), \qquad k \geqslant 3, \tag{4.3}$$

where $r = q+s-1$.

Proof. Without any loss of generality we assume $n \geqslant r+2$. We take $b \in (0, 1)$ so that

$$\frac{1}{4(s+1)\, bn^{2(r+1)}} - \frac{b^r}{4(r+2)!} = A,$$

and fix an arbitrary collection Y of points y_i such that $-1+b=y_1>y_2>\cdots>y_s>-1$. Set

$$Q_{r+2}(x):=(x-y_1)^{r+2};$$

$$f(x):=(x-y_1)_+^{r+2}:=\begin{cases} Q_{r+2}(x), & \text{if } \ x\geqslant -1+b, \\ 0, & \text{if } \ x<-1+b. \end{cases}$$

Obviously $f\in\mathbb{C}^{(r)}\cap\Delta^{(q)}(Y)$. For an arbitrary polynomial $p_n\in\mathbb{P}_n\cap\Delta^{(q)}(Y)$ put

$$R_n(x):=p_n(x)-Q_{r+2}(x)$$

and consider the divided difference $[y_1, ..., y_{s+1}; R_n^{(q)}]$, where $y_{s+1}:=-1$. Since $p_n\in\Delta^{(q)}(Y)$, then $p_n^{(q)}(y_i)=0$, $i=\overline{1, s}$, whence

$$[y_1, ..., y_{s+1}; p_n^{(q)}]=\frac{p_n^{(q)}(-1)}{\Pi(-1)}\geqslant 0.$$

Put

$$S(x):=\frac{(r+2)!}{(s+1)!}(x-y_{s+1})\,\Pi(x)$$

and note that

$$\begin{aligned} &S^{(s)}(x)-Q_{r+2}^{(q+s)}(x)\\ &\quad\equiv\frac{(r+2)!}{s+1}((y_1-y_2)+(y_1-y_3)+\cdots+(y_1-y_{s+1})). \end{aligned}$$

Therefore,

$$\begin{aligned} &-[y_1, ..., y_{s+1}; Q_{r+2}^{(q)}]\\ &\quad=[y_1, ..., y_{s+1}; s-Q_{r+1}^{(q)}]=\frac{1}{s!}(S^{(s)}(\theta)-Q_{r+2}^{(q+s)}(\theta))\\ &\quad=\frac{(r+2)!}{(s+1)!}((y_1-y_2)+(y_1-y_2)+\cdots+(y_1-y_{s+1}))\geqslant\frac{(r+2)!}{(s+1)!}b. \end{aligned}$$

Hence there exists a point $\theta\in(-1, -1+b)$ such that

$$R_n^{(r+1)}(\theta)=s![y_1, ..., y_{s+1}; R_n^{(q)}]\geqslant\frac{(r+2)!}{s+1}b.$$

Applying Markov inequality we get

$$\frac{(r+2)!}{s+1}\, b \leqslant R_n^{(r+1)}(\theta) \leqslant \|R_n\|\, n^{2(r+1)}$$
$$\leqslant (\|f-p_n\| + \|f-Q_{r+2}\|)\, n^{2(r+1)}$$
$$= (\|f-p_n\| + b^{r+2})\, n^{2(r+1)},$$

whence

$$\|f-p_n\| \geqslant \frac{b(r+2)!}{(s+1)\, n^{2(r+1)}} - b^{r+2}.$$

On the other hand,

$$\omega_3(f^{(r)}; 1) = \omega_3(f^{(r)} - Q_{r+2}^{(r)}; 1) \leqslant 8\, \|f^{(r)} - Q_{r+2}^{(r)}\| = 4b^2(r+2)!.$$

Therefore

$$\frac{\|f-p_n\|}{\omega_3(f^{(r)}; 1)} \geqslant \frac{1}{4b(s+1)\, n^{2(r+1)}} - \frac{b^r}{4(r+2)!} = A. \quad \blacksquare$$

Example 4.2, Example 4.2 and its Corollary lead to

LEMMA 4.1. *In the case of type "$\oplus$", "$\ominus$" and "$-$" Propositions $W(k, r, s, q)$ and $J(k, r, s, q)$ are false.*

4.2. *Cases "$-$"*

Everywhere below we will use the following notations. For a fixed collection $Y \in \mathbb{Y}_s$ put

$$\Pi_1(x) := \Pi_1(x; Y) := \prod_{i=2}^{s} (x - y_i) \qquad (= \Pi(x)/(x-y_1),\ x \neq y_1),$$
$$d := d(Y) := \tfrac{1}{2} \min\{1 - y_1,\ y_1 - y_2\},$$

if $s > 1$. If $s = 1$, then we put

$$\Pi_1(x) := 1, \qquad d := d(Y) := \tfrac{1}{2}(1 - |y_1|).$$

Put

$$M_0 := M_0(Y) := \|\Pi_1\|, \qquad M := M(Y) := \Pi_1(y_1)$$

and note,

$$0 < M \leqslant M_0 \leqslant 2^{s-1}.$$

EXAMPLE 4.3. For every n and $A>0$, and for each s, $Y\in\mathbb{Y}_s$, $k>s+1$ and q, there is a function $f(x)=f(x;q,k,n,Y,A)$ such that $f\in\Delta^{(q)}(Y)\cap\mathbb{C}^{(q-1)}$ and

$$E_n^{(q)}(f;Y)>A\omega_k(f^{(q-1)};1). \tag{4.4}$$

Proof. Without any loss of generality assume $n\geqslant k+q-2$. For a fixed $b\in(0,d)$ set

$$Q_{s+q}(x):=q_{s+q}(x;b):=\frac{1}{(q-1)!}\int_{y_1}^{x}(x-u)^{q-1}(u-y_1-b)\,\Pi_1(u)\,du;$$

$$(x-y_1-b)^*:=\begin{cases}0, & \text{if } x\in[y_1,y_1+b],\\ x-y_1-b, & \text{otherwise;}\end{cases}$$

$$g(x):=g(x;b):=\frac{1}{(q-1)!}\int_{y_1}^{x}(x-u)^{q-1}(y-y_1-b)^*\,\Pi_1(u)\,du.$$

Clearly, $g\in\Delta^{(q)}(Y)\cap\mathbb{C}^{(q-1)}$. For an arbitrary polynomial $p_n\in\mathbb{P}_n\cap\Delta^{(q)}(Y)$ put

$$r_n(x):=p_n(x)-Q_{s+q}(x)$$

and observe that

$$r_n^{(q)}(y_1)=-Q_{s+q}^{(q)}(y_1)=b\Pi_1(y_1)=bM. \tag{4.5}$$

Applying Markov inequality

$$\|r_n^{(q)}\|\leqslant n^{2q}\,\|r_n\|$$

we get

$$bM=r_n^{(q)}(y_1)\leqslant n^{2q}\,\|r_n\|,$$

whence

$$\begin{aligned}\frac{bM}{n^{2q}}&\leqslant\|r_n\|\leqslant\|p_n-g\|+\|g-Q_{s+q}\|\\ &\leqslant\|p_n-g\|+\frac{2^{q-1}M_0}{(q-1)!}\int_{y_1}^{y_1+b}(y_1+b-u)\,du\leqslant\|p_n-g\|+M_0b^2,\end{aligned}$$

i.e.

$$\|p_n-g\|\geqslant\frac{bM}{n^{2q}}-M_0b^2=\frac{bM}{n^{2q}}\left(1-\frac{M_0bn^{2q}}{M}\right). \tag{4.6}$$

On the other hand we have

$$\omega_k(g^{(q-1)}; 1) = \omega_k(g^{(q-1)} - Q_{s+q}^{(q-1)}; 1) \leqslant 2^k \, \|g^{(q-1)} - Q_{s+q}^{(q-1)}\|$$
$$= 2^k \int_{y_1}^{y_1+b} (b + y_1 - u)\, \Pi_1(u)\, du \leqslant 2^{k-1} M_0 b^2. \tag{4.7}$$

Now, in order to prove (4.4) we take

$$b_n := \frac{1}{2^k} \frac{Md}{M_0(A+1)} \left(\frac{1}{n}\right)^{2q}, \qquad f(x) := g(x; b_n),$$

and note that $b_n < d$. It follows from (4.6) and (4.7) that

$$\frac{\|p_n - f\|}{\omega_k(f^{(q-1)}; 1)} \geqslant \frac{b_n M}{n^{2q}} \left(1 - \frac{1}{2}\right) \frac{1}{2^{k-1} b_n^2 M_0} = \frac{A+1}{d} > A. \quad \blacksquare$$

Corollary. *For each s, q, $Y \in \mathbb{Y}_s$, $r < q$, $k > q + s - r$, n and $A > 0$ there is a function $f(x) = f(x; q, r, k, n, Y, A)$ such that $f \in \Delta^{(q)}(Y) \cap \mathbb{C}^{(r)}$ and*

$$E_n^{(q)}(f; Y) > A\omega_k(f^{(r)}; 1). \tag{4.8}$$

Indeed, for $r = q - 1$ such function is constructed in Example 4.3; for $r < q - 1$ one can take the same function and use the inequality $\omega_{k+r+1-q}(f^{(q-1)}; 1) \geqslant \omega_k(f^{(r)}; 1)$.

Example 4.4. For every n and $A > 0$, and for each s, $Y \in \mathbb{Y}_s$, $k > s + 2$ and q, there is a function $f(x) = f(x; q, k, n, Y, A)$ such that $f \in \Delta^{(q)}(Y) \cap \mathbb{C}^{(q)}$ and

$$E_n^{(q)}(f; Y) > A\omega_k(f^{(q)}; 1). \tag{4.9}$$

Proof. Without any loss of generality assume $n \geqslant k + q - 1$. For a fixed $b \in (0, d)$ set

$$Q_{s+q+2}(x) := Q_{s+q+2}(x; b)$$
$$:= \frac{1}{(q-1)!} \int_{y_1}^{x} (x-u)^{q-1} \, ((u - y_1)^2 - b^2)\, \Pi(u)\, du;$$

$$((x - y_1)^2 - b^2)_+ := \begin{cases} 0, & \text{if } (x - y_1)^2 \leqslant b^2 \\ (x - y_1)^2 - b^2, & \text{otherwise;} \end{cases}$$

$$g(x) =: g(x; b)$$
$$=: \frac{1}{(q-1)!} \int_{y_1}^{x} (x-u)^{q-1} \, ((u - y_1)^2 - b^2)_+ \, \Pi(u)\, du.$$

Clearly, $g \in \Delta^{(q)}(Y) \cap \mathbb{C}^{(q)}$. For an arbitrarily polynomial $p_n \in \mathbb{P}_n \cap \Delta^{(q)}(Y)$ put

$$r_n(x) := p_n(x) - Q_{s+q+2}(x).$$

Since $p_n \in \Delta^{(q)}(Y)$, then $p_n^{(q+1)}(y_1) \geqslant 0$, whence

$$\begin{aligned} r_n^{(q+1)}(y_1) &= p_n^{(q+1)}(y_1) - Q_{s+q+2}^{(q+1)}(y_1) \\ &\geqslant -Q_{s+q+2}^{(q+1)}(y_1) = b^2 \Pi'(y_1) = b^2 \Pi_1(y_1) = b^2 M. \end{aligned} \tag{4.10}$$

Applying Markov inequality

$$\|r_n^{(q+1)}\| \leqslant n^{2(q+1)} \|r_n\|,$$

we get

$$Mb^2 \leqslant \|r_n^{(q+1)}\| \leqslant n^{2(q+1)} \|r_n\|,$$

whence

$$\begin{aligned} \frac{Mb^2}{n^{2(q+1)}} &\leqslant \|r_n\| \leqslant \|p_n - g\| = \|g - Q_{s+q+2}\| \\ &\leqslant \|p_n - g\| + \frac{2^{q-1} M_0}{(q-1)!} \int_{y_1}^{y_1+b} (b^2 - (u-y_1)^2)(u-y_1)\, du \\ &< \|p_n - g\| + M_0 b^4, \end{aligned}$$

where we used the identity $\Pi(u) = (u-y_1)\, \Pi_1(u)$. Hence

$$\|p_n - g\| \geqslant \frac{Mb^2}{n^{2(q+1)}} - M_0 b^4 = \frac{Mb^2}{n^{2(q+1)}} \left(1 - \frac{M_0 b^2 n^{2(q+1)}}{M}\right). \tag{4.11}$$

On the other hand

$$\begin{aligned} \omega_k(g^{(q)}; t) &= \omega_k(g^{(q)} - Q_{s+q+2}^{(q)}; t) \\ &\leqslant 2^k \|g^{(q)} - Q_{s+q+2}^{(q)}\| < 2^{k-1} M_0 b^3. \end{aligned} \tag{4.12}$$

In order to prove (4.9) we take

$$b_n := \frac{Md}{M_0 2^k (A+1)} \left(\frac{1}{n}\right)^{2(q+1)}, \qquad f(x) := g(x; b_n),$$

and note that $b_n < d$. It follows from (4.11) and (4.12) that

$$\frac{\|p_n - f\|}{\omega_k(f^{(q)}; 1)} \geqslant \frac{b_n^2 M}{n^{2(q+1)}} \left(1 - \frac{1}{2}\right) \frac{1}{2^{k-1} b_n^3 M_0} = \frac{A+1}{d} > A. \quad \blacksquare$$

Example 4.4, Example 4.3 and its Corollary lead to

LEMMA 4.2. *In the cases of type "$-$" Propositions $W(k, r, s, q, Y)$ and $J(k, r, s, q, Y)$ are false for each $Y \in \mathbb{Y}_s$.*

Thus the proof of Theorem 2 is completed.

To end the proof of Theorem 4 we have to consider cases of type "$\ominus$".

4.3. *Cases "$\ominus$"*

Remark, we do not have the cases of type "$\ominus$" when $s = 1$.

EXAMPLE 4.5. For every n and for each $s \neq 1$, $Y \in \mathbb{Y}_s$, $k > 2$ and q, there is a function $f(x) := f(x; k, n, q, Y)$ such that $f \in \Delta^{(q)}(Y) \cap \mathbb{C}^{(q-1)}$ and

$$E_n^{(q)}(f; Y) > B_Y n^{(k/2)-1} \frac{1}{n^{q-1}} \omega_k\left(f^{(q-1)}; \frac{1}{n}\right). \tag{4.13}$$

Proof. We use the notation of Example 4.3 and repeat its arguments up to (4.5). Thus we have

$$r_n^{(q)}(y_1) = bM.$$

Using Dzyadyk inequality

$$r_n^{(q)}(y_1)\, \rho_n^q(y_1) \leqslant c\, \|r_n^{(q-1)} \rho_n^{q-1}\|,$$

and Leviatan inequality (2.4), we get

$$\begin{aligned}
bM\rho_n^q(y_1) &\leqslant c\, \|r_n^{(q-1)} \rho_n^{q-1}\| \\
&\leqslant c\, \|(p_n^{(q-1)} - g^{(q-1)})\, \rho_n^{q-1}\| + c\, \|(g^{(q-1)} - Q_{s+q}^{(q-1)})\, \rho_n^{q-1}\| \\
&\leqslant c\, \|p_n - g\| + \frac{c}{n^{q-1}} E_{n-q+1}(g^{(q-1)}) \\
&\quad + c\, \|\rho_n^{q-1}\|\, \|g^{(q-1)} - Q_{s+q}^{(q-1)}\| \\
&\leqslant c\, \|p_n - g\| + \frac{c}{n^{q-1}} \|g^{(q-1)} - Q_{s+q}^{(q-1)}\| \leqslant c\, \|p_n - g\| + \frac{cb^2}{n^{q-1}}.
\end{aligned}$$

On the other hand

$$\omega_k\left(g^{(q-1)};\frac{1}{n}\right)\leqslant\omega_k\left(g^{(q-1)}-Q^{(q-1)}_{s+q};\frac{1}{n}\right)+\omega_k\left(Q^{(q-1)}_{s+q};\frac{1}{n}\right)$$

$$\leqslant 2^k\,\|g^{(q-1)}-Q^{(q-1)}_{s+q}\|+\frac{1}{n^k}\,\|Q^{(q-1+k)}_{s+q}\|$$

$$\leqslant c\left(b^2+\frac{1}{n^k}\right).$$

Thus

$$\frac{\|p_n-g\|\,n^{q-1}}{\omega_k\left(g^{(q-1)};\dfrac{1}{n}\right)}\geqslant\frac{cbM\rho_n^q(y_1)\,n^{q-1}-c\left(b^2+\dfrac{1}{n^k}\right)}{c\left(b^2+\dfrac{1}{n^k}\right)}-c^*$$

$$>\frac{cbM(1-y^2)^{q/2}}{n\left(b^2+\dfrac{1}{n^k}\right)}-c^*=:4B_Y\frac{b}{n\left(b^2+\dfrac{1}{n^k}\right)}-c^*,$$

where we used the inequality $\rho_n(y_1)>\sqrt{1-y_1^2}/n$. Now, to prove (4.13) let us take

$$b_n:=\frac{1}{n^{k/2}},\qquad f(x):=g(x;b_n).$$

So we obtain

$$\frac{\|p_n-f\|\,n^{q-1}}{\omega_k\left(f^{(q-1)};\dfrac{1}{n}\right)}\geqslant 2B_Yn^{(k/2)-1}-c^*$$

$$=B_Yn^{(k/2)-1}\left(2-\frac{c^*}{B_Yn^{(k/2)-1}}\right)\geqslant B_Yn^{(k/2)-1}$$

for all $n\geqslant N:=N(Y)$, where the integer N is chosen so that

$$c^*\leqslant B_YN^{(k/2)-1},\qquad b_N<d,\qquad N\geqslant k+q-2.$$

Thus for $n\geqslant N(Y)$ the inequality (4.13) is proved. For $n<N(Y)$ (4.13) follows from the inequality $E_n^{(q)}(f;Y)\geqslant E_N^{(q)}(f;Y)$. ∎

COROLLARY. *For each $s \neq 1$, $Y \in \mathbb{Y}_s$, q, $r < q$, $k > q - r + 1$ and n there is a function $f(x) = f(x; q, k, r, n, Y)$ such that $f \in \Delta^{(q)}(Y) \cap \mathbb{C}^{(r)}$ and*

$$E_n^{(q)}(f; Y) > B_Y n^{(k+r-1-q)/2} \frac{1}{n^r} \omega_k\left(f^{(r)}; \frac{1}{n}\right) \geqslant B_Y \sqrt{n} \frac{1}{n^r} \omega_k\left(f^{(r)}; \frac{1}{n}\right).$$

Indeed, for $r = q - 1$ such function is constructed in Example 4.5; for $r < q - 1$ one can take the same function and use the inequality $t^{q-1-r} \omega_{k+r+1-q}(f^{(q-1)}; t) \geqslant \omega_k(f^{(r)}; t)$.

EXAMPLE 4.6. For every n and for each $s \neq 1$, $Y \in \mathbb{Y}_s$, $k > 3$ and q, there is a function $f(x) := f(x; Y, k, n, q)$ such that $f \in \Delta^{(q)}(Y) \cap \mathbb{C}^{(q)}$ and

$$E_n^{(q)}(f; Y) > B_Y n^{(k/3)-1} \frac{1}{n^q} \omega_k\left(f^{(q)}; \frac{1}{n}\right). \tag{4.16}$$

Proof. We use the notation of the Example 4.4 and repeat its argument up to (4.10). Thus we have

$$r_n^{(q+1)}(y_1) \geqslant b^2 M.$$

Using Dzyadyk inequality

$$r_n^{(q+1)}(y_1)\, \rho_n^{q+1}(y_1) \leqslant c\, \|r_n^{(q)} \rho_n^q\|,$$

and Leviatan inequality (2.4), we get

$$\begin{aligned}
b^2 M \rho_n^{q+1}(y_1) &\leqslant c\, \|r_n^{(q)} \rho_n^q\| \\
&\leqslant c\, \|(p_n^{(q)} - g^{(q)})\, \rho_n^q\| + c\, \|(g^{(q)} - Q_{s+q+2}^{(q)})\, \rho_n^q\| \\
&\leqslant c\, \|p_n - g\| + \frac{c}{n^q} E_{n-q}(g^{(q)}) + c\, \|\rho_n^q\|\, \|g^{(q)} - Q_{s+q+2}^{(q)}\| \\
&\leqslant c\, \|p_n - g\| + \frac{c}{n^q} \|g^{(q)} - Q_{s+q+2}^{(q)}\| \leqslant c\, \|p_n - g\| + \frac{cb^3}{n^q}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\omega_k\left(g^{(q)}; \frac{1}{n}\right) &\leqslant \omega_k\left(g^{(q)} - Q_{s+q+2}^{(q)}; \frac{1}{n}\right) + \omega_k\left(Q_{s+q+2}^{(q)}; \frac{1}{n}\right) \\
&\leqslant 2^{k-1} M_0 b^3 + \frac{1}{n^k} \|Q_{s+q+2}^{(q+k)}\| \leqslant c\left(b^3 + \frac{1}{n^k}\right).
\end{aligned}$$

Thus

$$\frac{\|p_n - g\|\, n^q}{\omega_k\left(f^{(q)}; \frac{1}{n}\right)} \geqslant \frac{cM\rho_n^{q+1}(y_1)\, b^2 n^q - c\left(b^3 + \frac{1}{n^k}\right)}{c\left(b^3 + \frac{1}{n^k}\right)}$$

$$=: \frac{cM\rho_n^{q+1}(y_1)\, b^2 n^q}{c\left(b^3 + \frac{1}{n^k}\right)} - c^*$$

$$> cM(1+y_1)^{(q+1)/2}\, \frac{b^2}{n\left(b^3 + \frac{1}{n^k}\right)} - c^*$$

$$=: 4B_Y \frac{b^2}{n\left(b^3 + \frac{1}{n^k}\right)} - c^*.$$

Now, in order to prove (4.14) we take

$$b_n := \frac{1}{n^{k/3}}, \qquad f(x) := g(x; b_n).$$

So we obtain

$$\frac{\|p_n - f\|}{\omega_k\left(f^{(q)}; \frac{1}{n}\right)} > 2B_Y n^{(k/3)-1} - c^*$$

$$= B_Y n^{(k/3)-1}\left(2 - \frac{c^*}{B_Y n^{(k/3)-1}}\right) \geqslant B_Y n^{(k/3)-1}$$

for all $n \geqslant N := N(Y)$, where the integer N is chosen so that

$$c^* \leqslant B_Y N^{(k/3)-1}, \qquad b_N < d, \qquad N \geqslant k+q-1.$$

Thus for $n \geqslant N(Y)$ the inequality (4.14) is proved. For $n < N(Y)$ (4.14) follows from the inequality $E_n^{(q)}(f;\, Y) \geqslant E_N^{(q)}(f;\, Y)$. ▮

Lemma 4.2, Example 4.6, Example 4.5 and its Corollary lead to

LEMMA 4.3. *In the cases of type "$\ominus$" and "$-$" Proposition $J(k, r, s, q, Y)$ is false for each $Y \in \mathbb{Y}_s$.*

Theorem 4 is proved.

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